U.C. MATH BOWL 2017

LEVEL I — Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Recall that $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$. Which is bigger

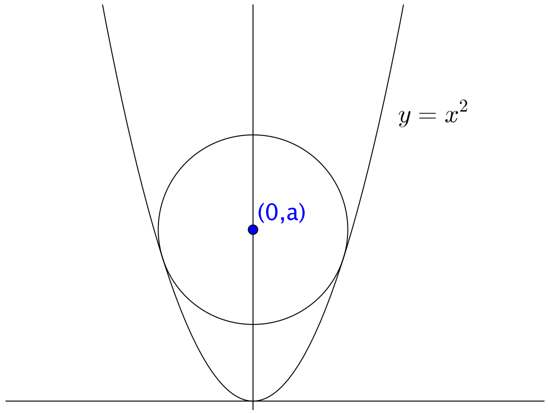
$$\sqrt[999]{999!} \quad \text{or} \quad \sqrt[1000]{1000!}?$$

Answer: $\sqrt[1000]{1000!}$

Solution: Raising both numbers to the power 999000 we see we must only compare $a = 1000!^{999}$ and $b = 999!^{1000}$.

$$\begin{aligned} ab &= \frac{1000!^{999}}{999!^{999}} \times \frac{1}{999!} \\ &= 1000^{999} \times \frac{1}{999!} \\ &= \frac{1000}{999}\frac{1000}{998}\cdots\frac{1000}{2}\frac{1000}{1} \\ &> 1 \cdot 1 \cdots 1 \cdot 1 \\ &= 1. \end{aligned}$$

2. The figure shows the graph of the parabola $y = x^2$ and a circle with center $(0, a)$ that is tangent to the parabola.



For each $a > 0$ determine the radius of the circle with center $(0, a)$ that is tangent to the parabola.

Answer:

$$r(a) = \begin{cases} a & \text{if } 0 < a \le 1/2 \\ \sqrt{a - 1/4} & \text{if } a \ge 1/2 \end{cases}$$

Solution: A circle and the parabola are tangent at a point they have in common if their tangent lines there coincide.

The key idea of the solution is to consider a point (b, b^2) with $b \neq 0$ on the parabola and note that the normal to the parabola (with slope $-1/(2b)$ has y-intercept $a = b^2 + 1/2$. This shows that for $a \ge 1/2$ the circle with center $(0, a)$ is tangent to the parabola at (b, b^2). The radius of this circle satisfies

$$r^2 = (b - 0)^2 + (b^2 - a)^2 = a - 1/4.$$

For smaller values of a the circle with center $(0, a)$ and radius a is tangent to the parabola at $(0, 0)$.

3. The line ℓ is tangent at $(-1, -2)$ to a circle centered at $(0, 0)$. What is the x intercept of ℓ?

Tangents to circles are perpendicular to radial segments so ℓ has slope $-1/2$. To move from $(-1, -2)$ to the x-axis ℓ must move up 2 units and that says it must move left 4 units. This brings it to $x = -5$.

4. A certain type of toy brick comes in two sizes: 2 cm by 4 cm, and 2 cm by 2 cm. The 2 x 4 bricks cost 11 cents, the 2 x 2 bricks cost 5 cents. If you are restricted to a maximum of 200 of each type of brick, what is the largest area you can cover with $16 worth of bricks? How many of each type of brick would be used?

Say we use x bricks that are 2×2 and y bricks that are 2×4. The cost is

$$C = 5x + 11y$$

and the area covered is $4x + 8y$. So the problem is to maximize

$$F = 4x + 8y$$

subject to the requirements that x and y are non-negative integers and

$$\begin{aligned} x &\leq 200 \\ y &\leq 200 \\ 5x + 11y &\leq 1600 \end{aligned}$$

Since $5(-2) + 11(1) = 1$ we see that $5(-3200) + 11(1600) = 1600$ thus the integer points (x, y) we consider are

$$\begin{aligned} x &= -3200 + 11k \\ y &= 1600 - 5k \end{aligned}$$

for integers k that ensure $200 \geq x \geq 0$ and $200 \geq y \geq 0$.

From

$$0 \leq 11k - 3200 \leq 200$$

we conclude that

$$\frac{3200}{11} \leq k \leq \frac{3400}{11}$$

while from

$$0 \leq 1600 - 5k \leq 200$$

we conclude that

$$\frac{1400}{5} \leq k \leq \frac{1600}{5}$$

Combining these inequalities we see that we must restrict k to be an integer that satisfies

$$\max\left(\frac{1400}{5}, \frac{3200}{11}\right) \leq k \leq \min\left(\frac{3400}{11}, \frac{1600}{5}\right)$$

or,

$$291 \leq k \leq 309.$$

The area covered is then

$$A = 4(-3200 + 11k) + 8(1600 - 5k) = 4k.$$

This area is largest when $k = 309$ and that corresponds to an area of 1236 and

$$x = -3200 + 11k = 199$$
$$y = 1600 - 5k = 55.$$

5. Find constants a and b so that, for $0 < x < \pi/12$,

$$\frac{\sin(9x)}{\cos(3x)} + \frac{\cos(9x)}{\sin(3x)} = a\cot(bx)$$

Answer: $a = 2$ and $b = 6$.

Solution: get a common denominator. Use double angle formula for sin and difference of angles formula for cos.