U.C. MATH BOWL 2017

LEVEL II — Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. If 1/5 and 7/15 are the first and fifth terms of an arithmetic sequence, what is the sum the second, third, and fourth terms?

> If d is the common difference between successive terms of the sequence, then the second, third, and fourth terms can be represented as $1/5+d$, $1/5+2d$, and $1/5+3d$. We know that $1/5+4d = 7/15$ so $d = 1/15$. The sum of the three terms is $3(1/5)+6(1/15) = 1$.

2. Suppose you pick an integer at random between 1 and 100 inclusive. What is the chance it is the sum of exactly two square numbers?

> The table below shows that there are 43 integers in the range $1, \ldots, 100$ that are the sum of two perfect squares. Only a few numbers can be written as a sum of two squares in more than one way.

	0	1	4	9	16	25	36	49	64	81	100
0	0	1	4	9	16	**25**	36	49	64	81	100
1		2	5	10	17	26	37	**50**	**65**	82	*101*
4			8	13	20	29	40	53	68	**85**	*104*
9				18	**25**	34	45	58	73	90	*109*
16					32	41	52	**65**	80	97	*116*
25						**50**	61	74	89	*106*	*125*
36							72	**85**	**100**	*117*	*136*
49								98	*113*	*130*	*149*
64									*128*	*145*	*164*
81										*162*	*181*
100											*200*

> The chance of picking one of these numbers at random is 43%.

3. The calculation

$$(3+\sqrt{2})^2 = 11+6\sqrt{2}$$

shows that the number $11+6\sqrt{2}$ has a square root with the form $a+b\sqrt{2}$ where a and b are integers. Show that the number

$$19+6\sqrt{2}$$

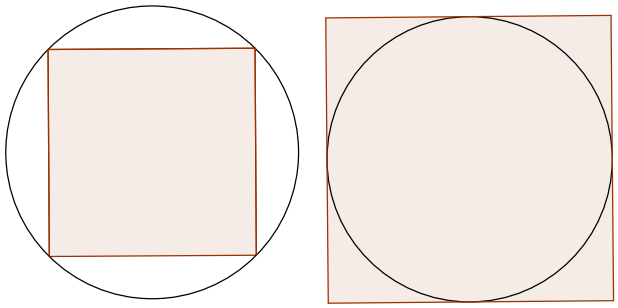
also has a square root of this form.

Guess that the root is $a + b\sqrt{2}$. Then, squaring,

$$19 + 6\sqrt{2} = (a^2 + 2b^2) + 2ab\sqrt{2}.$$

Inspection shows that $a = 1$ and $b = 3$ is a solution.

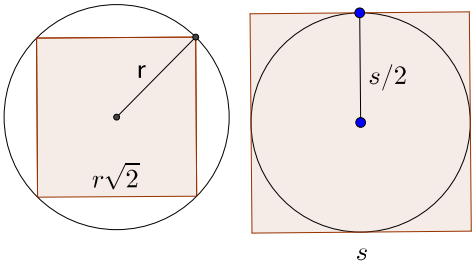
4. What's a better fit, a round peg in a square hole or a square peg in a round hole?



(a) Find the ratio of the area of a circle to the area of the square in which it is inscribed.

(b) Find the area of a square to the area of the circle in which it is inscribed.

(c) Which of these ratios is larger?



On the right in the figure the ratio of the areas is

$$\frac{\pi(s/2)^2}{s^2} = \pi/4.$$

On the left the ratio is

$$\frac{2r^2}{\pi r^2} = 2/\pi.$$

Then, since $\pi > 3$, we note that

$$\frac{\pi}{4} > \frac{3}{4} > \frac{2}{3} > \frac{2}{\pi}.$$

So a round peg in a square hole is a better fit!

5. Find a positive two digit number N with the property that 3 times the number minus 4 times the number with the digits reversed is 19. For example, if $N = 27$ then the number with the digits reversed is 72.

Write the number as $10t + u$ where t and u are the ten's and unit's digits of the number. We require

$$3(10t + u) - 4(10u + t) = 19$$

which says

$$-37u + 26t = 19.$$

We see that u must be odd since otherwise the left side of the equation is even and can't equal 19. Trying $u = 1, 3, 5, \ldots$ we find that $u = 3$

allows $t = 5$. So one such number is 53. Checking other possible values of u shows that this is actually the only such number positive number. There are other numbers with the main property but they're negative.