U.C. MATH BOWL 2017

LEVEL I — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. The polynomial $2x^3 + 3x^2 + 5x + 2$ has two roots x_1 and x_2 that are complex numbers — they are not real numbers. What is the sum $x_1 + x_2$?

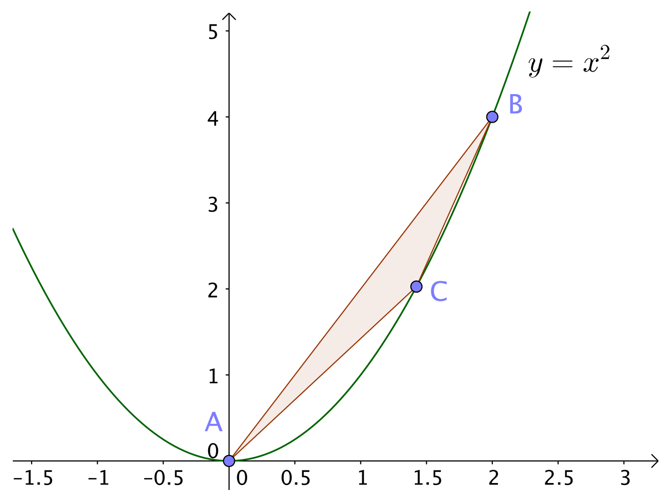
> The only possible rational roots of the polynomial are $\pm 1, \pm 2, \pm 1/2$. A bit of checking, aided by Descartes' rule of signs, perhaps, shows that $-1/2$ is a root. Since the sum of all the roots is $-3/2$ we see that $x_1 + x_2 = -1$.

2. Evaluate the limit

$$\lim_{x\to\infty} \frac{1}{x^4} \int_0^{x^2} \frac{t^4}{1+t^3}\,dt$$

> L'Hospital's Rule, the Fundamental Theorem, and the chain rule let us write the equivalent limit
>
> $$\lim_{x\to\infty} \frac{2x\frac{x^8}{1+x^6}}{4x^3} = \frac{1}{2}\lim_{x\to\infty} \frac{x^9}{x^3+x^9} = 1/2$$

3. Find the point C on the graph of $y = x^2$ between the points $A(0, 0)$ and $B(2, 4)$ so that the area of $\triangle ABC$ is as large as possible.



Unless the tangent to the parabola at C is parallel to the segment $\overline{AB}$ it is possible to move C on the graph so as to increase the distance from C to $\overline{AB}$ and thus increase the area of the triangle.

Since the area of the triangle is proportional to the distance from C to $\overline{AB}$, the area is maximized with the tangent at $C = (c, c^2)$ has slope 2. That says $c = 1$ and $C = (1, 1)$.

4. Consider the array of consecutive odd numbers (each row beyond the first contains 2 more numbers than the previous row):

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & 5 & 7 & & \\ & 9 & 11 & 13 & 15 & 17 & \\ 19 & 21 & 23 & 25 & 27 & 29 & 31 \\ & & & \vdots & & & \end{array}$$

In what row will the number 2017 appear?

The first k rows contain the first k^2 odd integers. We notice this easily looking at the table but the general fact that the sum of consecutive odds makes squares is pretty well know.

Since 2017 is the $\frac{2017-1}{2} + 1 = 1009$-th odd integer we need a row number n so that

$$(n-1)^2 < 1009 \le n^2.$$

Since $32^2 = 2^{10} = 1024$, the number 2017 appears in the 32-nd row.

5. The arithmetic sequence $a_1, a_2, a_3, \ldots, a_n$ has a common difference of 10 and the sum $a_1 + a_2 + \cdots + a_{100} = 2017$. Determine the value of the sum of even terms:

$$S = a_2 + a_4 + \cdots + a_{100}.$$

We use that fact that

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ s_n &= a_1 + \cdots a_n = (n/2)(a_1 + a_n) \end{aligned}$$

and the information we're given to determine a_1 and a_{100}. That is, we solve the two equations

$$\begin{aligned} a_{100} &= a_1 + 99(10) \\ 2017 &= \frac{100}{2}(a_1 + a_{100}) \end{aligned}$$

to determine that

$$\begin{aligned} a_1 &= -474.83 \\ a_{100} &= 515.17 \end{aligned}$$

Then $a_2 = -474.83 + 10 = -464.83$ and the sum we want is the sum of 50 terms of an arithmetic starting at -464.83 and ending with 515.17. That is,

$$S = \frac{50}{2}(515.17 - 464.83) = 1258.5.$$