

# U.C. MATH BOWL 2017

LEVEL III — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. In the addition shown each letter represents a distinct digit. Can you find ways to assign digits to letters so that the addition is correct?

$$\begin{array}{r} \text{O N E} \\ + \text{O N E} \\ \hline \text{T W O} \end{array}$$

$271 + 271 = 542$  works. Many people notice that O must be even and this cuts down on the guessing required especially since O must be less than 5. Otherwise there'd be a carry and the sum would be 4 digits instead of 3.

2. The mean annual income of 10 people is \$40,000. Four of the ten get equal pay raises. The recalculated mean annual income is now \$43,000. What was the amount of each raise?

Answer: \$7500.

Solution: If the mean annual income for the 10 before the raises is \$40,000, then the 10 earned a combined income of  $10 \times \$40,000 = \$400,000$ . Now, 4 received a pay raise of some unknown amount. Call the amount  $a$ . The new combined income for the 10 after the raise is equal to  $\$400,000 + 4a$ . The mean income after the raise is  $(\$400,000 + 4a)/10 = \$43,000$ . Solving for  $a$  yields \$7500.

3. Ben runs around the track at a pace of 1 lap every 80 seconds. Sally runs on the track in the opposite direction. The two meet every 30 seconds. How long does it take Sally to run each lap around the track?

Answer: 48 seconds.

Solution: Consider what happens between one time that Ben and Sally meet and the next. During this period Ben runs 30 (seconds)/

$80 \text{ (seconds/lap)} = 3/8 \text{ of a lap. So Sally runs } 5/8 \text{ lap in the same time. Her speed is therefore } 5/8 \text{ (laps)/ } 30 \text{ (seconds) which amounts to } 1 \text{ (lap)/ } 48 \text{ (seconds).}$

4. Consider all the triangles having all three sides of integer length. Are there more such triangles with perimeter 11 or with perimeter 12? Justify your answer.

This can be answered by simply enumerating the possibilities. The key point is that the side lengths must satisfy the triangle inequality with each being less than the sum of the other two. Said another way, segments of lengths  $a, b, c$  form a triangle exactly when each is less than half the sum of three.

There are 4 triangles with perimeter 11 and 3 with perimeter 12. For both perimeters, the longest the side of any triangle can be is 5.

For perimeter 11 we notice that the longest side possible is length 5. The shortest side can't be length 4 or more since otherwise the other two sides would have combined length of no more than 7 which is impossible if each is of length at least 4. That leaves just these triangles: (1, 5, 5); (2, 4, 5); (3, 3, 5); (3, 4, 4).

For perimeter 12 the longest possible side length is 5. The shortest side can't have length more than 4 (argue as above) or less than 2 (A side of length 1 and two sides of maximal length 5 yields a perimeter of just 11). That leaves just these triangles: (2, 5, 5); (3, 4, 5); (4, 4, 4)

5. There is an imaginary island called the Island of Knights and Knaves. On this island, there are people called knights, who always tell the truth, and people called knaves, who always lie. The two types are indistinguishable by sight. On a vacation to this island, you meet two inhabitants: Bill and Bob. Bob says "We are both the same kind, either both knights or both knaves", but Bill says "We are both different kinds. One of us is a knave and the other is a knight" Is Bob a knight or a knave?

Answer: Bob is a knave and Bill is a knight.

Solution: Suppose Bob is a knight. Then he is telling the truth, and this means Bill is a knight, and Bill is telling the truth. But if Bill is a knight, Bill actually telling a lie by saying they are different kinds! This can't be!

Suppose Bob is knave. When he says they are both the same kind, he is lying, which means that Bill must be a knight. If Bill is a knight,

he is telling the truth when he says they are different kinds.