U.C. MATH BOWL 2020

LEVEL I— Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

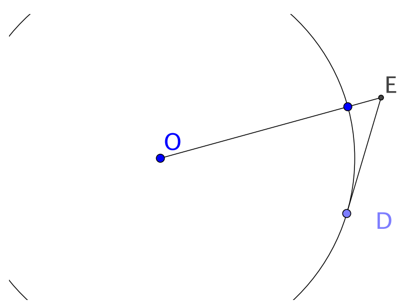
1. You're a thief, and you've managed to break into the vault of an bank that holds 100 sacks of coins. One of the sacks contains gold coins, while the other 99 are filled with counterfeit coins. You cannot tell the difference between the gold coins and the fakes by handling the coins, looking at them, biting them, or testing them. The fake coins weigh exactly 1 ounce each, while the real gold coins weigh 1.01 ounces. There is a large scale with enough room for all the sacks in the vault, but as soon as you weigh something it will trigger an alarm, so you can use the scale just once before you must flee the vault and bank. How can you figure out which sack of coins contains the real gold by only weighing something on the scale once?

Number the sacks. Take n coins from sack number n and weigh the collection of coins so obtained. If the weight is $5050 + K/100$ the real coins are in bag number K.

2. Explain why, for every integer m, it must be that $m^9 - m$ is a multiple of 10.

$\phi(10) = 4$. Most approaches consider the results of raising the digits $1, \ldots, 9$ to the 9th power. In each case, the right most digit is unchanged by this procedure. So the right digit of $m^9 - m$ is 0.

3. Suppose you're standing on top of a light house so your eyes are 100 feet above the level of a calm sea. You look out to where the sea and sky meet. About how far is that horizon? Give your answer in miles and provide some estimate of your accuracy. You can assume that the Earth is a sphere with radius 4000 miles. There 5280 feet in a mile.



Compute in miles. 100 feet is $h = 100/5280$ miles. Let R be the radius of the Earth and D the distance to the horizon. Since tangents to circles are orthogonal to radii,

$$(R + h)^2 = D^2 + R^2.$$

This says

$$R^2 + 2hR + h^2 = D^2 + R^2$$

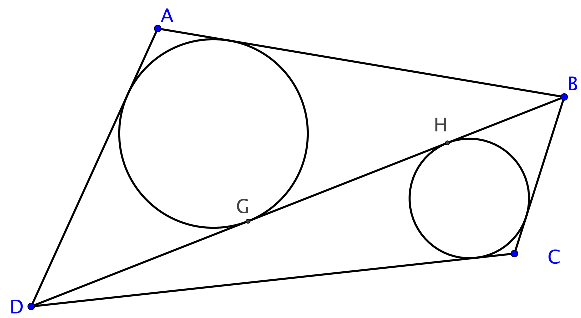
so that

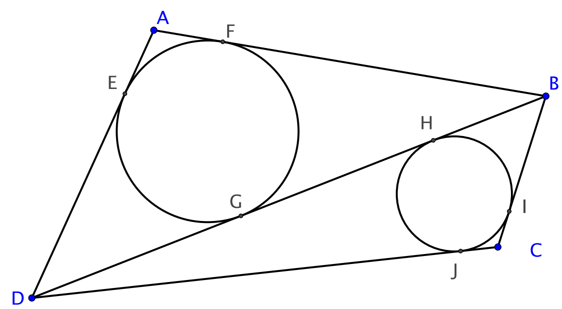
$$D = \sqrt{h^2 + 2hR} \approx \sqrt{2hR}\sqrt{1 + \frac{h}{2R}}.$$

This amounts to about 12.31 miles correct to 1 decimal place.

Who knows what approximations people will make? Probably none if they're using a calculator. There's lots of reasonable approaches and most are better than just using a calculator and saying nothing. The typical approximation is for $x \approx 0$ so that $\sqrt{1+x} = 1 + x/2$

4. The convex quadrilateral $ABCD$ has sides$|AB| = 2$, $|BC| = 8$, $|CD| = 6$ and $|DA| = 7$. It is divided into two triangles by the diagonal $\overline{AC}$. Circles are inscribed in these two triangles touching the diagonal at points G and H. Find the distance $|GH|$.





$$AE = AF$$
$$BF = BG$$
$$BH = BI$$
$$CI = CJ$$
$$DJ = DH$$
$$DC = DE$$

Also

$$AF + FB = AB$$
$$BI + IC = BC$$
$$CJ + JD = CD$$
$$DE + EA = AD$$

$GH = HD - DG = DJ - DE$ $GH = GB - HB = BF - BII$

Solve some equations.

There's a problem with the labelling of the segments in the figure; they lead to $|FG| = -3.5$.

The rubric used was 0 points if no use of the key geometric idea. 5 if the idea was indicated but not used effectively, 10 if good progress towards the solution.

5. Let $f(x) = x|x|^p$ where $p > 0$ is some real number. Find the derivative $f'(x)$. If possible, write your answer as a single formula.

$$\boxed{f'(x) = (p+1)|x|^p.}$$

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LEVEL I — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

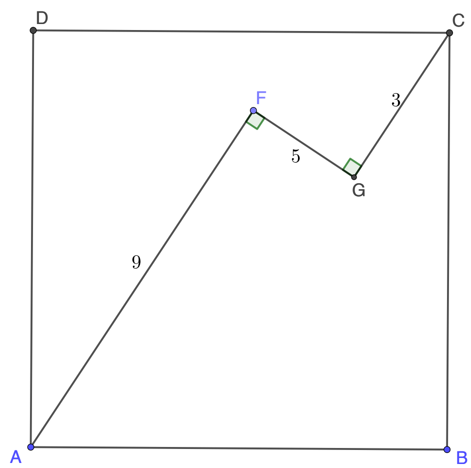
1. Suppose that

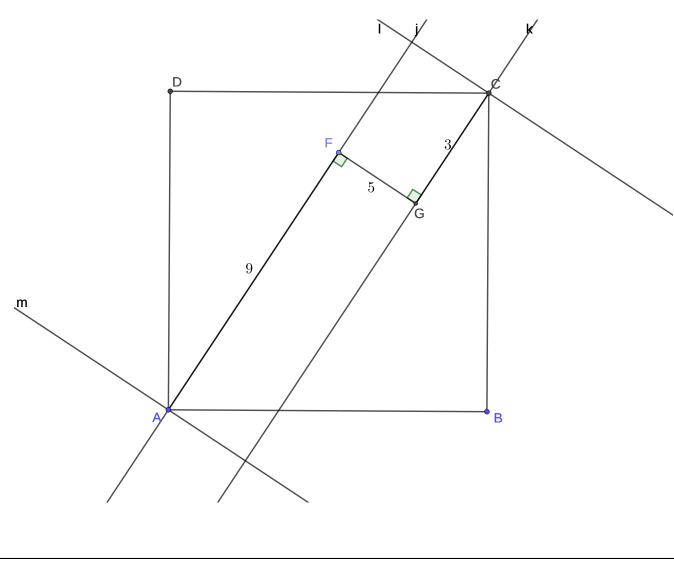
$$\ln(x) + \ln(y) = 9$$
$$\ln(x) - \ln(y) = 5$$

What is $\log_y(x)$?

$\ln(x) = 7$ and $\ln(y) = 2$ so $\log_y(x) = 7/2$.

2. In the figure, $ABCD$ is a square and points F and G lie inside the square so that $|AF| = 9$, $|CG| = 3$, $|FG| = 5$ and $\overline{AF} \perp \overline{FG}$ and $\overline{FG} \perp \overline{CG}$. Find the area of square $ABCD$.





3. Find the equation of a line ℓ so that the line together with the curve $y = x^2$ bounds a region of area $A > 0$.

If we use the line $y = a$ the area bounded is

$$2\int_0^a \sqrt{y}\, dy = \frac{4}{3}y^{3/2}|_0^a = \frac{4}{3}a^{3/2}/$$

So we can take

$$a = (\frac{3}{4}A)^{2/3}.$$

4. Suppose f is a function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x\to 0}\frac{f(x)}{x} = 1.$$

- Find $f(0)$

Plugging in $x = 0$ and $y = 0$, we obtain $f(0) = 2f(0)$, thus $f(0) = 0$.

- Find $f'(0)$.

We have

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{f(x)}{x} = 1.$$

- Find $f'(x)$.

We have

$$f'(x) = \lim_{y\to 0}\frac{f(x+y) - f(x)}{y} = \lim_{y\to 0}\frac{f(y) + x^2y + xy^2}{y} = 1 + x^2$$

- Find $f(x)$.

$$f(x) = f(0) + \int_0^x f'(s)\, ds = x + \frac{x^3}{3}.$$

5. Tangent lines T_1 and T_2 are drawn at the points P_1 and P_2, respectively, on the parabola $y = x^2$, and they intersect at a point P. Another tangent line is drawn at a point between P_1 and P_2; it intersects T_1 at a point Q_1 and T_2 at a point Q_2. Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$

Most easily handled as a geometry problem using info about tangents to parabolas. Can also be calculated explicitly using coordinates, the derivative, etc.