

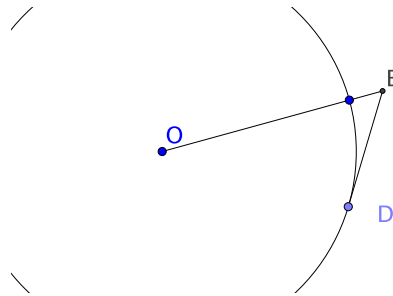
# U.C. MATH BOWL 2020

## LEVEL I— Session 1

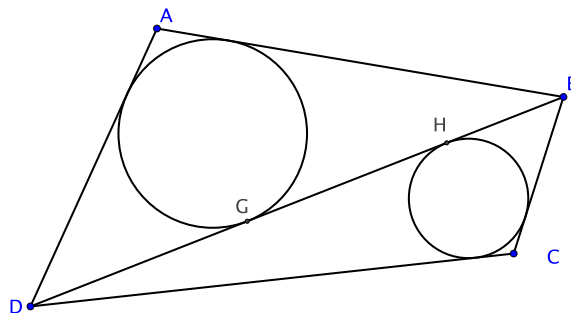
Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. You're a thief, and you've managed to break into the vault of a bank that holds 100 sacks of coins. One of the sacks contains gold coins, while the other 99 are filled with counterfeit coins. You cannot tell the difference between the gold coins and the fakes by handling the coins, looking at them, biting them, or testing them. The fake coins weigh exactly 1 ounce each, while the real gold coins weigh 1.01 ounces. There is a large scale with enough room for all the sacks in the vault, but as soon as you weigh something it will trigger an alarm, so you can use the scale just once before you must flee the vault and bank. How can you figure out which sack of coins contains the real gold by only weighing something on the scale once?
2. Explain why, for every integer  $m$ , it must be that  $m^9 - m$  is a multiple of 10.
3. Suppose you're standing on top of a light house so your eyes are 100 feet above the level of a calm sea. You look out to where the sea and sky meet. About how far is that horizon? Give your answer in miles and provide some estimate of your accuracy. You can assume that the Earth is a sphere with radius 4000 miles. There 5280 feet in a mile.



4. The convex quadrilateral  $ABCD$  has sides  $|AB| = 2$ ,  $|BC| = 8$ ,  $|CD| = 6$  and  $|DA| = 7$ . It is divided into two triangles by the diagonal  $\overline{AC}$ . Circles are inscribed in these two triangles touching the diagonal at points  $G$  and  $H$ . Find the distance  $|GH|$ .



5. Let  $f(x) = x|x|^p$  where  $p > 0$  is some real number. Find the derivative  $f'(x)$ . If possible, write your answer as a single formula.

# U.C. MATH BOWL 2020

## LEVEL I — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

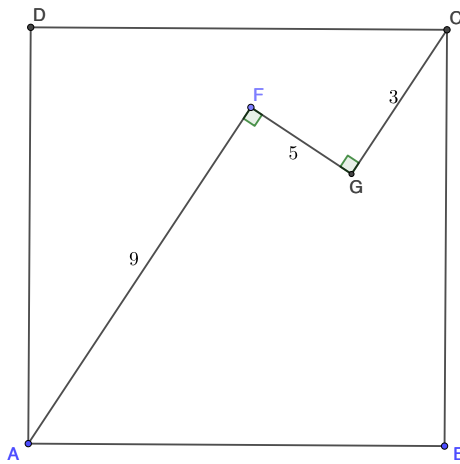
1. Suppose that

$$\ln(x) + \ln(y) = 9$$

$$\ln(x) - \ln(y) = 5$$

What is  $\log_y(x)$ ?

2. In the figure,  $ABCD$  is a square and points  $F$  and  $G$  lie inside the square so that  $|AF| = 9$ ,  $|CG| = 3$ ,  $|FG| = 5$  and  $\overline{AF} \perp \overline{FG}$  and  $\overline{FG} \perp \overline{CG}$ . Find the area of square  $ABCD$ .



3. Find the equation of a line  $\ell$  so that the line together with the curve  $y = x^2$  bounds a region of area  $A > 0$ .
4. Suppose  $f$  is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers  $x$  and  $y$ . Suppose also that

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1.$$

- Find  $f(0)$
  - Find  $f'(0)$ .
  - Find  $f'(x)$ .
  - Find  $f(x)$ .
5. Tangent lines  $T_1$  and  $T_2$  are drawn at the points  $P_1$  and  $P_2$ , respectively, on the parabola  $y = x^2$ , and they intersect at a point  $P$ . Another tangent line is drawn at a point between  $P_1$  and  $P_2$ ; it intersects  $T_1$  at a point  $Q_1$  and  $T_2$  at a point  $Q_2$ . Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$