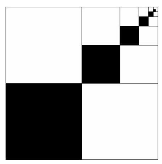
U.C. MATH BOWL 2020

LEVEL II — Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

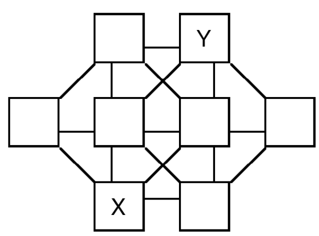
1. Assuming that the indicated pattern continues on forever what fraction of the original square is colored black? 

If f is the fraction then $f = 1/3$ since $f = 1/4 + f/4$.

Alternatively one could sum a geometric series (this is what most teams did) since the largest filled square is 1/4 of the entire square and each subsequent filled square is 1/4 the area of the previous one.

$$\sum_{1}^{\infty} \frac{1}{4^n} = \frac{1}{4}\frac{1}{1-\frac{1}{4}} = \frac{1}{3}.$$

2. In the diagram shown, the boxes are to be filled with the digits 1 through 8 (each used exactly once). No two boxes connected directly by a line segment can contain consecutive digits. What is the sum of the digits in the boxes marked X and Y?



Try filling in the boxes! Youll quickly discover that the two central squares must be the ones with 1 and 8 in them since these squares have 6 neighbors. And then youll see that the squares on the ends must be filled with 7 and 2. After that, theres two places in which 3 can go. Having chosen that, theres only one place 4 can go that leaves places for 5 and 6. Any way you place 3, the boxes X and Y end up with the digits 4 and 5 (or 6 and 3). The sum is 9 either way.

3. You sit at a sprawling table with a pile of thousands of quarters in front of you, but you don't know exactly how many. You have a blindfold on, so you cannot see the quarters, but you do know that exactly 20 quarters are tails-side-up, and the rest are heads-up.You can move the quarters and flip them over as much as you want, but remember, you cannot see what you are doing. Though you can feel the quarters, you cannot determine which side is heads and which side is tails just by touch. How do you separate the quarters into two groups that have the same number of tails-side-up quarters in them?

Take 20 of the quarters from the big pile and turn them over. If the 20 you pick include k with tails up, then there are $20 - k$ tails up coins in what remains of the original collection. After you turn over your 20 coins, there are $20 - k$ coins in your collection that are tails up.

4. Suppose a and b are positive numbers with $a < b$. Arrange the following in increasing order:

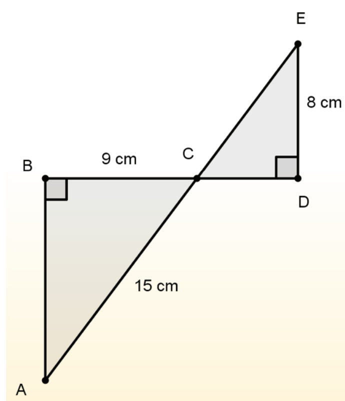
$$\frac{b+1}{a+1}, \frac{b+1}{a}, \frac{b}{a}, \frac{b}{a+2}, \frac{b}{a+1}$$

Some are easy: $\frac{b}{a+2} < \frac{b}{a+1} < \frac{b}{a} < \frac{b+1}{a}$ based on same number of parts and same sized arts comparisons.

Same number of parts also shows $(b+1)/(a+1) < (b+1)/a$.

So the only real question is the location of $(b+1)/(a+1)$.

5. In the figure line BD intersects AE at the point C. The length of BC is 9 cm, then length of AC is 15 cm and the length of ED is 8 cm. Find the combined area of triangle ABC and triangle DEC.



$|AB| = 12$ and $|CD| = 6$ The combined area is $54 + 24 = 787$ square centimeters.

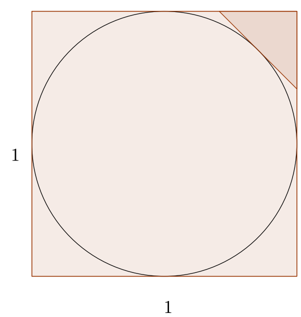
U.C. MATH BOWL 2020

LEVEL II — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. A circle is inscribed in a square with side length 1. In each corner of the square there is a right isosceles triangle that shares a vertex angle with the square and has hypotenuse tangent to the circle. What is the area of one of these triangles?



> $(3-2\sqrt{2})/4$ The height of the triangle above its hypotenuse is $(\sqrt{2}-1)/2$ since it is half the difference of the diagonal of the square and the diameter of the circle. The area of the triangle is the square of this altitude since the altitude is half the hypotenuse.
>
> $$\left((\sqrt{2}-1)/2\right)^2 = \frac{1}{4}(3-2\sqrt{3}).$$

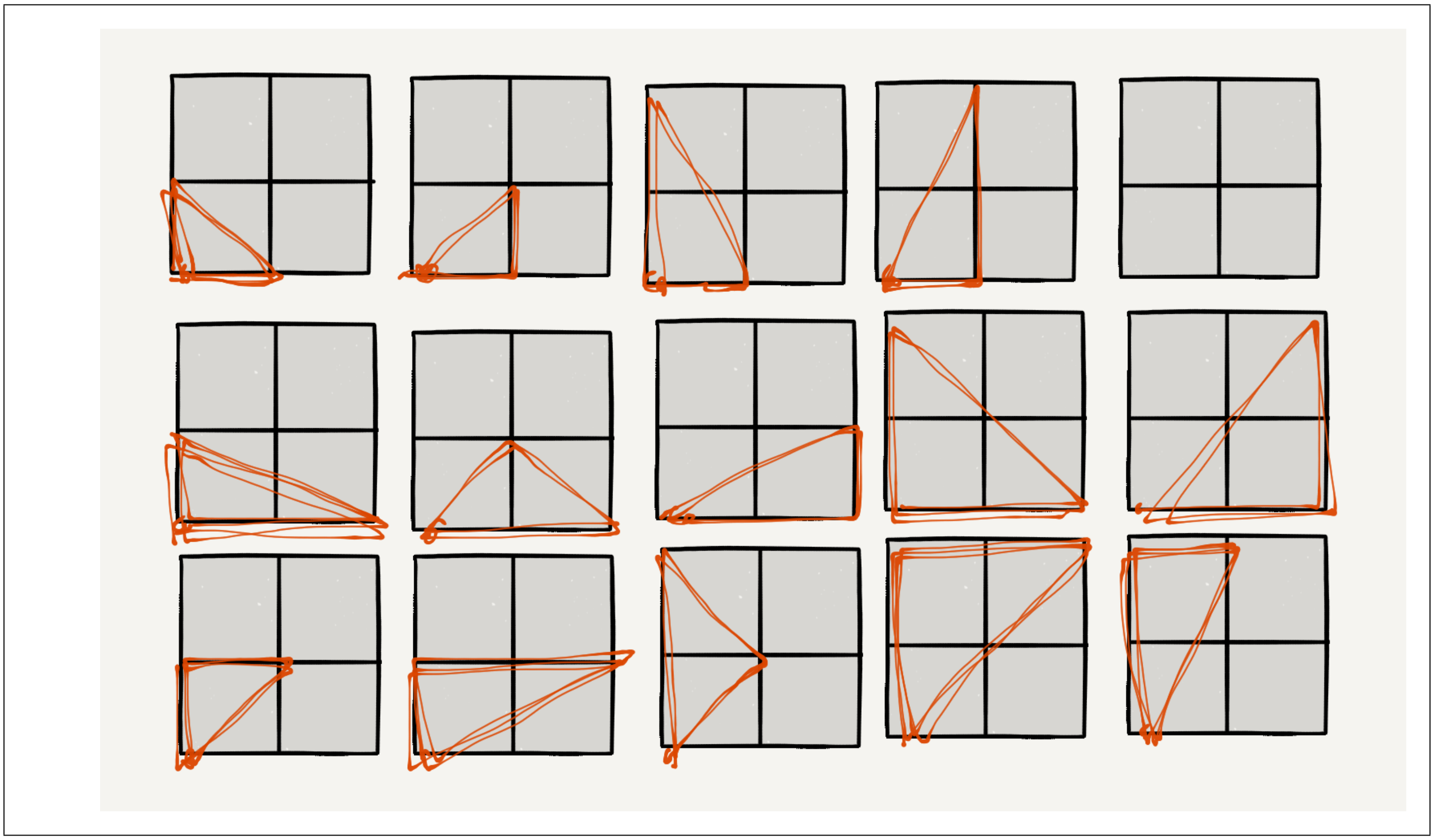
2. A bag originally contained 12 marbles in some combination of red, green, and blue. After 2 marbles are removed from the bag the probabilities of drawing a marble of different colors satisfy: $P(\text{draw green}) < P(\text{draw red}) < P(\text{draw blue})$.

   Is it possible that the original bag contained more green marbles than blue? What about more green marbles than red marbles?

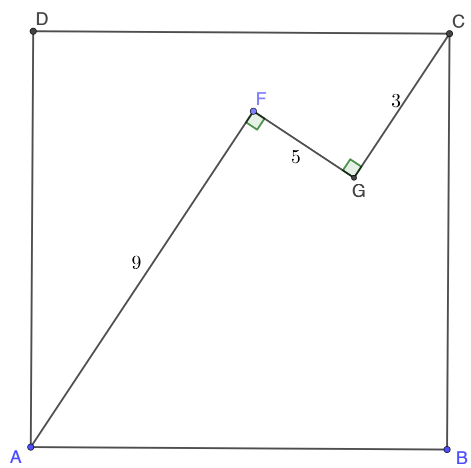
> With 2 marbles removed from the bag, it contains 10 marbles. The largest possible number of green marbles is then 2 with $P(G) = 2/10$, $P(R) = 3/10$, $P(B) = 5/10$). So the original bag might have had as many as 4 green marbles — not as many as the blue marbles, but more than the number of red marbles.

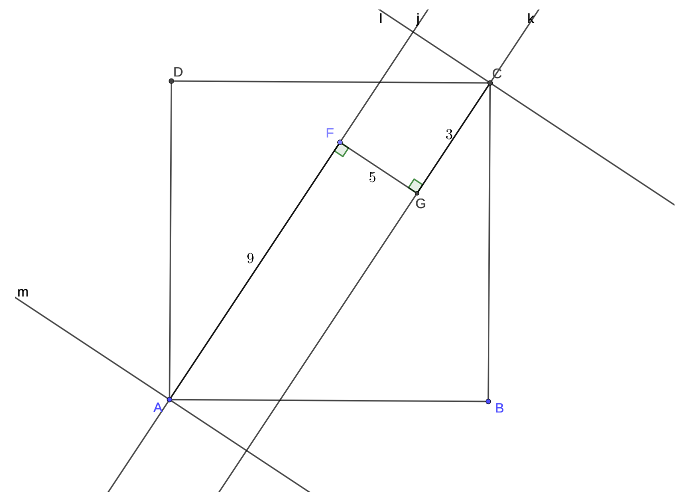
3. Suppose that $O(0,0)$, $P(a,b)$ and $Q(c,d)$ are three points with integer coordinates between 0 and 2 inclusive. So $0 \le a,b,c,d \le 2$. In how many ways can you pick a,b,c,d so that $\triangle OPQ$ is a right triangle with one of its angles measuring 90°?

> People get slightly different answers depending on whether they're counting the different right triangles (14) or counting the different right triangles and then allowing that P and Q can be interchanged (28).
>
> Most people find these 14 triangles. To be systematic, consider all possible locations of P and list the places where Q may go.



4. In the figure, $ABCD$ is a square and points F and G lie inside the square so that $|AF| = 9$, $|CG| = 3$, $|FG| = 5$ and $\overline{AF} \perp \overline{FG}$ and $\overline{FG} \perp \overline{CG}$. Find the area of square $ABCD$.





5. A school has 40 teachers and each teacher teaches 4 classes. Each class has 30 students and 1 teacher. Each student takes 5 classes. How many students are there at the school?

There's 4×40 classes and so 30×160 student class-periods. With 5 classes per student that makes $4800/5 = 960$ students.