U.C. MATH BOWL 2021

LEVEL I

There are 7 questions for you work on in this Math Bowl. Each is printed on a separate page.

Write your school, team number, and the names of the team members on the first page.

Write your school and team number on each question's page.

All your work and answers to each question should go on that question's page (or you can use extra pages if you need more room). Please only put the answer to one question on each page.

Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. I have 17 matching pairs of earrings in a drawer all jumbled together. How many earrings must I remove from the drawer at random — without looking — in order to guarantee that I have at least four matching pairs?

You could be unlucky enough to pick 17 earrings without ever getting a single pair. But then the next 4 earrings you draw will certainly match 4 that you've already drawn and make up the required 4 pairs.

2. A function f defined for real numbers x is called *convex* if for all real numbers x and y we have
$$f(x) \leq \frac{f(x+y)+f(x-y)}{2}.$$
Show that if f is convex, $f(0)=0$, and $f(1)=1$, then $f(2^n) \geq 2^n$ for all integers $n \geq 1$.

Using the definition with $x=1$ and $y=1$, we have
$$1 = f(1) \leq \frac{f(2)+f(0)}{2} = \frac{f(2)}{2} \qquad \Longrightarrow \qquad f(2) \geq 2.$$
Let's now assume the statement holds for $n=k$ and show it for $n=k+1$. To that end, use the definition with $x=2^k$ and $y=2^k$:
$$2^k \leq f(2^k) \leq \frac{f(2^k+2^k)+f(0)}{2} = \frac{f(2^{k+1})}{2} \qquad \Longrightarrow \qquad f(2^{k+1}) \geq 2^{k+1}.$$
By induction, the statement is proved.

3. Show that
$$\frac{1}{17} \leq \int_1^2 \frac{1}{1+x^4}\,dx \leq \frac{7}{24}$$

On the interval $[1,2]$ we have

$$\frac{1}{1+2^4} \le \frac{1}{1+x^4} \le \frac{1}{x^4},$$

thus,

$$\int_1^2 \frac{1}{17}\,dx \le \int_1^2 \frac{1}{1+x^4}\,dx \le \int_1^2 \frac{1}{x^4}\,dx$$

The left-hand side is $\frac{1}{17}$; the right-hand side is $\frac{7}{24}$. Of course, much better estimates are possible.

BTW, you can actually calculate the required anti-derivative! $1+x^4$ factors as a product of two distinct, real irreducible quadratics so the anti-derivative can be obtained based on a partial fractions expansion. The anti-derivative will involve the arctan and log. In fact,

$$\begin{aligned}\int (1+x^4)^{-1}\,dx &= \frac{1}{4\sqrt{2}}\left(-\log\left(x^2-\sqrt{2}x+1\right)+\log\left(x^2+\sqrt{2}x+1\right)\right)\\ &+\frac{1}{4\sqrt{2}}\left(-2\arctan\left(1-\sqrt{2}x\right)+2\arctan\left(\sqrt{2}x+1\right)\right)\end{aligned}$$

If you adopt this approach you still are faced with the problem of evaluating these expressions with the accuracy required to establish the estimates requested. Though, perhaps, it would be reasonable to simply rely on a scientific calculator.

4. The Chebyshev polynomials T_n are defined by $T_n(x) = \cos(n \arccos(x))$, $n = 0, 1, 2, 3, \ldots$

   (a) Compute $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$.

$$T_0(x) = \cos(0) = 1$$

$$T_1(x) = \cos(\arccos(x)) = x.$$

Finding T_2 and T_3 is much easier after having completed part (c) below!

   (b) Establish (prove) the trig identity that says

$$\cos(a+b) = 2\cos(a)\cos(b) - \cos(a-b)$$

Add the angle sum/difference identities

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

and

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b).$$

That gives you what you need.

(c) Show that for $n \geq 1$,

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

We've said $T_n(x) = \cos(n\theta)$ where $\cos(\theta) = x$. The previous part lets us write

$$\begin{aligned} T_{n+1}(x) &= \cos\left((n+1)\theta\right) \\ &= \cos(\theta + n\theta) \\ &= 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta) \\ &= 2xT_n(x) - T_{n-1}(x) \end{aligned}$$

5. A *quadratic trinomial* is a polynomial that has the form $p(x) = a + bx + cx^2$ for some constants a, b, c.

(a) Is it possible to find three quadratic trinomials such that each of them has a real zero, yet the sum of any two of them has no real zeros?

Yes. Here's an example:

$$(x-10)^2, x^2, (x+10)^2$$

(b) Is it possible to find three quadratic trinomials such that each of them has two distinct real zeros, yet the sum of any two of them has no real zeros?

Yes. Modify the example from part (a):

$$(x-10)^2 - 1, x^2 - 1, (x+10)^2 - 1$$

To come up with such an example, we make each of the polynomials in part (a) to have two zeros by subtracting small enough number from the perfect square.

6. Suppose $\log_{\sqrt{3}}(a) = \log_9(ab)$. What is $\log_a(b)$?

If we write $x = \log_{\sqrt{3}}(a)$ we're told $3^{x/2} = a$ and $9^x = 3^{2x} = ab$.

So

$$b = \frac{3^{2x}}{a} = \frac{3^{2x}}{3^{x/2}} = 3^{3x/2} = (3^{x/2})^3 = a^3.$$

So $\log_a(b) = 3$.

7. Let a, b, c be real numbers from the interval $(0, \pi/2)$ such that $\cos(a) = a$, $2\sin(\cos b) = b$ and $\cos(\sin c) = c$. Arrange the numbers a, b, c in order from smallest to largest.

Part of this problem is pretty easy, the other tougher and more open ended.

First, we show that $c > a$ by contradiction. Suppose $c < a$ then

$$\sin(c) < c < a$$

so that

$$c = \cos(\sin(c)) < \cos(a) = a,$$

a contradiction.

A similar argument shows that if d satisfies $d = \sin(\cos(d))$ then $d < a$. Since otherwise $d > a$ and we would have $\cos d < \cos a$ so that

$$\begin{aligned} d &= \sin(\cos(d)) \\ &< \sin(\cos(a)) \\ &= \sin(a) \\ &< a. \end{aligned}$$

The number b asked about in the problem, for which $b = 2\sin(\cos(b))$, satisfies $b > c$ and there seems to be no simple way to demonstrate this. One approach is to carry out numerical approximations to the roots of the equations defining c and b. Given the difficulty of the problem doing this even with a calculator seems like a good plan. Another possibility is to estimate $b - c$ and $b - a$ by using the mean value theorem; this seems to require using 2nd derivatives to establish the required estimates.