U.C. MATH BOWL 2021

LEVEL I

There are 7 questions for you work on in this Math Bowl. Each is printed on a separate page.

Write your school, team number, and the names of the team members on the first page.

Write your school and team number on each question's page.

All your work and answers to each question should go on that question's page (or you can use extra pages if you need more room). Please only put the answer to one question on each page.

Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. I have 17 matching pairs of earrings in a drawer all jumbled together. How many earrings must I remove from the drawer at random — without looking — in order to guarantee that I have at least four matching pairs?

2. A function f defined for real numbers x is called *convex* if for all real numbers x and y we have
$$f(x) \leq \frac{f(x+y)+f(x-y)}{2}.$$
Show that if f is convex, $f(0) = 0$, and $f(1) = 1$, then $f(2^n) \geq 2^n$ for all integers $n \geq 1$.

3. Show that
$$\frac{1}{17} \leq \int_1^2 \frac{1}{1+x^4}\,dx \leq \frac{7}{24}$$

4. The Chebyshev polynomials T_n are defined by $T_n(x) = \cos(n \arccos(x))$, $n = 0, 1, 2, 3, ...$
   (a) Compute $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$.
   (b) Establish (prove) the trig identity that says
   $$\cos(a+b) = 2\cos(a)\cos(b) - \cos(a-b)$$
   (c) Show that for $n \geq 1$,
   $$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

5. A *quadratic trinomial* is a polynomial that has the form $p(x) = a+bx+cx^2$ for some constants a, b, c.
   (a) Is it possible to find three quadratic trinomials such that each of them has a real zero, yet the sum of any two of them has no real zeros?
   (b) Is it possible to find three quadratic trinomials such that each of them has two distinct real zeros, yet the sum of any two of them has no real zeros?

6. Suppose $\log_{\sqrt{3}}(a) = \log_9(ab)$. What is $\log_a(b)$?

7. Let a, b, c be real numbers from the interval $(0, \pi/2)$ such that $\cos(a) = a$, $2\sin(\cos b) = b$ and $\cos(\sin c) = c$. Arrange the numbers a, b, c in order from smallest to largest.