U.C. MATH BOWL 2021

LEVEL II

There are 7 questions for you work on in this Math Bowl. Each is printed on a separate page.

Write your school, team number, and the names of the team members on the first page.

Write your school and team number on each question's page.

All your work and answers to each question should go on that question's page (or you can use extra pages if you need more room). Please only put the answer to one question on each page.

Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Can the sum of four consecutive integers be ever divisible by 4 (i.e. a multiple of 4)? Explain.

If the first of the four integers is a then the sum of the four consecutive integers is

$$a + (a + 1) + a + 2) + (a + 3) = 4a + 6$$

Dividing this sum by 4 always leaves a remainder of 2:

$$4a + 6 = 4(a + 1) + 2$$

so the sum of four consecutive integers always has a remainder of 2 when we divide it by 4.

2. Fifteen children have 100 lollipops among them. Can you explain why at least two children have the same number of lollipops?

If the kids each had a different number of lollipops then the smallest number they could have between them is

$$0 + 1 + 2 + \cdots + 14 = \frac{14 \cdot 15}{2} = 105.$$

So if they have 100 between them, they can't each have a different number of candies.

3. Which is greater, the sum of all even numbers from 0 to 1000 or the sum of all odd numbers from 1 to 999? By how much? (Hint: you shouldn't use a calculator.)

The sum of the even numbers is

$$S_e = 0 + 2 + 4 + \cdots + 998 + 1000$$

The sum of the odd ones is

$$S_o = (2(0) + 1) + (2(1) + 1) + (2(2) + 1) + \cdots + (2(499) + 1)$$

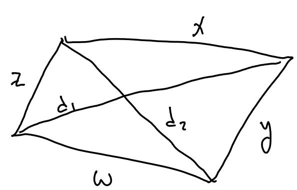
Except for the 1000, each of the 500 terms in the sum S_o is one larger than one of the terms in the sum S_e.

Therefore

$$S_e - S_0 = 1000 - 500 = 500.$$

4. Given a convex quadrilateral, which is bigger the sum of the lengths of its sides or the sum of the lengths of its diagonals? What if the quadrilateral isn't convex? (Convex means that both the diagonal are inside the quadrilateral.)

Repeated application of the triangle inequality shows that the perimeter exceeds the sum of the diagonal's lengths:



$x+y > d_2$ The triangle inequality!

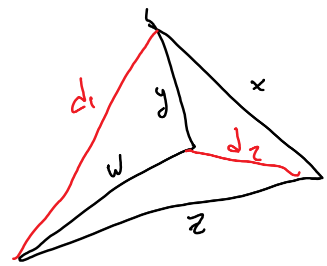
$y+w > d_1$

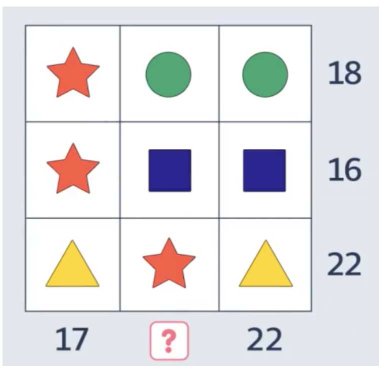
$w+z > d_2$

$z+x > d_1$

$2(x+y+w+z) > 2(d_1+d_2)$

People may think differently about diagonals in non-convex polygons and any reasonable approach is fine. As illustrated below, we reach the same conclusion if we agree that the diagonals of a quadrilateral are the segments connecting alternate vertices.





5. In the picture above, the same symbols represent the same numbers, and the numbers at the end of each row and column represent the sum of all numbers in that row or column. What goes into the question mark?

Write x, y, w, z for the numbers represented by the star, circle, square, and triangle respectively. The margin sums then require:

$$\begin{aligned} x + 2y &= 18 \\ x + 2w &= 16 \\ x + 2z &= 22 \\ 2x + z &= 17 \\ y + w + z &= 22 \end{aligned}$$

and we want the value of $x+y+w$. We *could* solve these simultaneous equation in hopes of determining the values of each of the variables x, y, w and then calculate the required sum. If we do this we obtain the solution

$$(x, y, w, z) = (4, 7, 6, 9),$$

showing that $x + y + w = 17$.

Easier, in this case, it would seem, is to use the first two equations to substitute for $y+w$:

$$\begin{aligned} x + y + w &= x + \frac{1}{2}(18 - x) + \frac{1}{2}(16 - x) \\ &= 17. \end{aligned}$$

6. Find all the numbers k for which $p(x) = x^3 - kx^2 + kx + 2$ has $x - 2$ as a factor.

The polynomial root theorem says that a is a root of a polynomial $q(x)$ exactly when $q(a) = 0$. Therefore the numbers k that make $p(x) = x^3 - kx^2 + kx + 2$ have $(x - 2)$ as a factor are the ones for which $p(2) = 0$. That says

$$p(2) = 8 - 4k + 2k + 2 = 0.$$

This is a linear equation that you can solve for a single value, 5 for k.

7. A right triangle has area 36 and perimeter P. How long is the triangle's hypotenuse h? (Your answer will depend on the perimeter P.)

Suppose the legs have lengths a and b. Then

$$\begin{aligned} a + b + h &= P && \text{perimeter} \\ ab &= 2(36) && \text{area} \\ a^2 + b^2 &= h^2. && \text{Pythagoras} \end{aligned}$$

There's lots of ways to proceed. Here's one.

Write the first equation as $a + b = P - h$ and square both sides to conclude that

$$a^2 + b^2 + 2ab = P^2 - 2Ph + h^2$$

Since Pythagoras tells us $a^2 + b^2 = h^2$ we can write this equation as

$$2ab = P^2 - 2Ph$$

and then the area equation tells us that $2ab = 4(36)$. We conclude that

$$2Ph = P^2 - 4(36)$$

and that's a linear equation we can solve to determine the value of h that corresponds to each perimeter P:

$$h = \frac{P^2 - 144}{2P}.$$