U.C. MATH BOWL 2021

LEVEL III

There are 7 questions for you work on in this Math Bowl. Each is printed on a separate page.

Write your school, team number, and the names of the team members on the first page.

Write your school and team number on each question's page.

All your work and answers to each question should go on that question's page (or you can use extra pages if you need more room). Please only put the answer to one question on each page.

Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Simplify the fraction. **Do not** use a calculator.

$$\frac{1 \cdot 2 \cdot 3 + 2 \cdot 4 \cdot 6 + 4 \cdot 8 \cdot 12 + 5 \cdot 10 \cdot 15}{1 \cdot 3 \cdot 5 + 2 \cdot 6 \cdot 10 + 4 \cdot 12 \cdot 20 + 5 \cdot 15 \cdot 25}$$

Each summand in the top is a multiple of $1 \cdot 2 \cdot 3$ and each summand in the bottom is a multiple of $1 \cdot 3 \cdot 5$ so the expression can be rewritten as

$$\frac{6(1 + 2^3 + 4^3 + 5^3)}{15(1 + 2^3 + 4^3 + 5^3)} = \frac{6}{16} = \frac{2}{5}.$$

2. An ice cream shop has 15 different flavors of ice cream and four different toppings. If a small sundae consists of one scoop of ice cream and your choice of two different toppings, how many different possible small sundaes are there?

There are six different ways to choose two toppings from the four possibilities. Then there are $15 \times 6 = 90$ different possible sundaes.

3. Arrange the numbers 2^{1000}, 3^{600} and 10^{300} in order from smallest to largest. Explain how you arrived at your answer.

The order (relative sizes) of the numbers and their 100-th roots is the same. So we only need to order the numbers $2^{10} = 1024$, 3^6, and $10^3 = 1000$. At this point one *could* do some mental arithmetic. More gratifying is to note that

$$3^6 = (3^2)^3 = 9^3 < 10^3$$

so that

$$9^3 < 1000 < 1024.$$

This tells us, taking 100-th powers, that

$$3^{600} < 10^{300} < 2^{1000}.$$

4. The mean annual income of 10 people is $40,000. Four of the ten get equal pay raises. The recalculated mean income is now $43,000. What was the amount of the pay raises?

If S is the sum of the pre-raise salaries and R is the amount of the raise that four were given we're told

$$\begin{aligned} \frac{S}{10} &= 40000 \\ \frac{S+4R}{10} &= 43000 \end{aligned}$$

Subtracting one of these equations from another we discover that

$$\frac{4R}{10} = 3000$$

showing that

$$R = \frac{30000}{4}.$$

5. How many cubic inches of dirt are removed in digging a hole that is 1 yard wide, 7 feet long, and 22 inches deep?

A tricky version of this question asks how many cubic inches of dirt are in such a hole — and the answer is 0!

Here we'll convert all the distances to inches

$$\begin{aligned} 1 \text{ yard} &= 3 \text{ feet} \\ &= 3 \text{ feet } \times 12 \ \frac{\text{in}}{\text{ft}} \\ &= 36 \text{ in} \\ 7 \text{ feet} &= 7 \text{ feet } \times 12 \ \frac{\text{in}}{\text{ft}} \\ &= 84 \text{ in} \end{aligned}$$

The volume of the hole is then

$$36 \times 84 \times 22 \text{ cubic inches},$$

or,

$$66{,}528 \text{ in}^3.$$

6. Can you divide the whole numbers from 1 up to 10 into two groups so that the sums of the numbers in the groups are the same? Show how to do this or explain why it can't be done.

It is impossible. If you could, and the sum of the numbers in each group were S, then

$$2S = 1 + 2 + 3 + \cdots + 10 = 55.$$

But this is impossible since 55 is odd and $2S$, whatever S is, is even.

7. A five-digit integer n includes the following digits: $0, 1, 2, 2, 2$. It is known that n is an perfect square. Find n.

Since it is a perfect square n can't have the 0 or a 2 in the 1's place. That only leaves three possibilities to check: 20221, 22021 and 22201. Only one of these is a perfect square: $149^2 = 22201$ is the prefect square.

An interesting question to contemplate in this context is how many different 5-digit numbers can you write down using the indicated digits?