U.C. MATH BOWL 2024

LEVEL I— Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Suppose that $x + 1/x = 1$. What is $x^{2024} + (1/x)^{2024}$?

We calculate that

$$(x^n + x^{-n})(x + x^{-1}) = (x^{n+1} + x^{-(n+1)})(x^{n-1} + x^{-(n-1)}).$$

So if we set $a_n = x^n + x^{-n}$ then

$$a_{n+1} = a_n - a_{n-1}.$$

We're told $a_1 = 1$. We can calculate that $1 = (x + 1/x)^2 = x^2 + x^{-2} + 2$ showing that $a_2 = -1$. Then we obtain the values of n_n using the recursion formula

n	1	2	3	4	5	6	7	8
a_n	1	-1	-2	-1	1	2	1	-1

The sequence a_n is periodic and the value of a_n only depends on the remainder of n when divided by 6.

Since $2024 = 6(339) + 1$ we see $x^{2024} + x^{-2024} = 1$.

An alternative approach is to write $x = (1 \pm i\sqrt{3})/2$ and use the fact that x is a cube root of -1.

2. Find all positive integers n such that $n = 12S(n)$, where $S(n)$ is the sum of the digits of n.

Assume n has k digits, $a_1, a_2, ..., a_k$. Then the equation to be solved becomes

$$a_1 + 10a_2 + 100a_3 + ... + 10^{k-1}a_k = 12(a_1 + a_2 + a_3 + ... + a_k),$$

with $a_k > 0$. Observe that the largest k we can have is $k = 3$. Indeed, the left-hand side is no smaller than 10^{k-1}, while the right-hand side is no larger than $12 \cdot 9k = 108k$. Since for $k > 3$, we clearly have $10^{k-1} > 108k$, we only need to consider the cases $k = 1$, $k = 2$, and $k = 3$.

If $k = 1$, the equation becomes $a_1 = 12a_1$ which is impossible for $a_1 > 0$. If $k = 2$, we get $a_1 + 10a_2 = 12a_1 + 12a_2$ or $-11a_1 = 2a_2$. This is again impossible for $a_1 \geq 0$ and $a_2 > 0$. Finally, if $k = 3$, we get

$$a_1 + 10a_2 + 100a_3 = 12(a_1 + a_2 + a_3) \quad \Longrightarrow \quad 88a_3 = 11a_1 + 2a_2,$$

where $a_3 > 0$. If $a_3 \geq 2$, this equation has no solution, because the left-hand side is at least $88 \cdot 2 = 172$, while the right-hand side is at most $10 \cdot 9 + 2 \cdot 9 = 108$. Therefore, we must have $a_3 = 1$ and so $88 = 11a_1 + 2a_2$. The only solution is $a_1 = 8$, $a_2 = 0$. Therefore, the original equation has only one solution: $\boxed{n = 108}$.

3. Which is larger 2023^{2024} or 2024^{2023}? How do you know?

2023^{2024} is larger. It is easier to compare $2023^{1/2023}$ with $2024^{1/2024}$ since $x^{1/x}$ is decreasing on $e \le x < \infty$.

This is because

$$\frac{d}{dx}x^{1/x} = x^{1/x}\frac{1 - \log x}{x^2},$$

which is negative when $\log x > 1$ (i.e. $x > e$).

4. Suppose that

$$\frac{d}{dx}f(3x) = 6x.$$

What is $f'(x)$?

Set $u = 3x$ so that $\frac{d}{dx}f(u) = 3f'(u) = 2u$ showing that $f'(u) = (2u/3)$.

5. Find all the integers n that can be written as

$$n = \frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{20}{a_{20}}$$

for some choice of positive integers

$$a_1 < a_2 < \cdots < a_{20}.$$

The inequalities on the a_i imply that each term has $\frac{k}{a_k} \le 1$. So the only possible integers are n in the range $1 \le n \le 20$.

Given an n in this range we choose

$$a_1 = 1,\ a_2 = 2,\ \cdots, a_{n-1} = n - 1$$

so that

$$\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{n-1}{a_{n-1}} = n - 1.$$

We now explain how to select the remaining a_i so that

$$\frac{n}{a_n} + \cdots + \frac{20}{a_{20}} = 1.$$

Take $a_k = k(21 - n)$ for $n \le k \le 20$ then

$$\frac{n}{a_n} + \cdots + \frac{20}{a_{20}} = \frac{1}{21-n} + \frac{1}{21-n} + \cdots + \frac{1}{21-n}.$$

Since there are $21 - n$ terms in this sum, each of size $1/(21 - n)$ the sum is 1 and so each n with $1 \le n \le 20$ can be written as

$$n = \frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{20}{a_{20}}.$$

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LEVEL I— Session 2

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Have Fun!

1. Let $\mathcal{C}$ be the "unit" circle in the xy-plane with equation $x^2+y^2 = 1$. Except for $(x, y) = (0, 0)$ we say that *the reflection* of the point (x, y) in $\mathcal{C}$ is the point

$$R(x, y) = (x/(x^2 + y^2), y/(x^2 + y^2)).$$

(a) Show that $R(R(x, y)) = (x, y)$.

Just calculate the composition. Stuff cancels.

(b) Describe the curve made up of the reflections of all the points on the line with equation $L(x, y) = y + x - 1 = 0$.

The equation for the reflection of the curve is $L(R^{-1}(x, y)) = 0$ which says that (x, y) is the reflection of a point that lies on the line. This equation is the same thing as $L(R(x, y)) = 0$. That says

$$\frac{y}{x^2 + y^2} + \frac{x}{x^2 + y^2} - 1 = 0.$$

We can write this as

$$x^2 - x + y^2 - y = 0$$

which is easier to understand after completing the squares:

$$(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}.$$

This describes a circle with center $(1/2, 1/2)$ and radius $1/\sqrt{2}$.

Technically there's one point on the circle — $(0, 0)$ — that isn't on the reflection of the line.

2. Every point on the plane is given one of two colors, red and blue. Show that there exists an isosceles triangle whose two equal sides have length 1 and whose vertices are all of the same color.

> Take any point in the plane and call it A. Let's say A is red. Draw a circle of radius 1 centered at A. If all points on the circle are blue, then fix any of them, say B, and take two points on the circle each a distance of 1 from B. That gives a desired isosceles triangle with three blue vertices.
>
> If there's a red point on the first circle, say C, draw a new circle of radius 1 centered at C (note that this circle will pass through A). If there's a red point on the circle besides A, we get a desired isosceles triangle with three red vertices. If A is the only red point on that circle, then clearly there are three blue points on it, two of which are a distance 1 from C, which gives a triangle with blue vertices.

3. Concerning fractions:

   (a) Show that for any two positive numbers a and b, we have $\frac{a}{b} + \frac{b}{a} \geq 2$.

> We have
> $$\frac{a}{b} + \frac{b}{a} \geq 2 \quad \Longleftrightarrow \quad a^2 + b^2 \geq 2ab \quad \Longleftrightarrow \quad (a-b)^2 \geq 0,$$
> which is a true statement.

   (b) Use the result of part (a) to show that for any three positive numbers a, b, and c, we have $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$.

> We have
> $$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{a}{c} + \frac{c}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right).$$
> By part (a), each of the expressions in parentheses is no less than 2, so the sum will be no less than $3 + 2 + 2 + 2 = 9$.

4. The obtuse triangle $\triangle ABC$ has sides of integer lengths $n, n+1, n+2$.

   (a) What is the value of n? Explain. Hint: there's only one such n.

> See below

   (b) What is the cosine of this triangle's largest angle?

> Obtuse means $n^2 + (n+1)^2 < (n+2)^2$. This says $-1 < n < 3$. But n must be positive and $n = 1$ violates the triangle inequality so the only possibility is $n = 2$.

Then the law of cosines says (where θ is the angle opposite the longest side and hence the obtuse angle)

$$4^2 = 2^2 + 3^2 - 2 \cdot (2 \cdot 3) \cos\theta,$$

showing that $\cos\theta = -1/4$.

5. Peter was 10 the day before yesterday. Next year, he'll be 13. Explain how these statements can both be true.

If today is 1 Jan 2024 then two days ago was 30 Dec 2023 and Peter was 10 years old. If his birthday is 31 Dec, then he turned 11 yesterday. This year, on 31 Dec 2024 he'll turn 12. And next year, in 2025, he'll be 13 on 31 Dec.