U.C. MATH BOWL 2024

LEVEL II — Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Find integers $x \le y \le z$ so that
$$2^x + 2^y + 2^z = 200$$
Hint: $2^x + 2^y + 2^z = 2^x(1 + 2^{y-z} + 2^{z-x})$.

$$\begin{aligned}
2^x(1 + 2^{y-x} + 2^{z-x}) &= 8 \cdot 25 \\
x &= 3 \\
2^{y-3} + 2^{z-3} &= 24 \\
2^{y-3}(1 + 2^{z-y}) &= 8 \cdot 3 \\
y &= 6 \\
1 + 2z - 6 &= 3 \\
z &= 7
\end{aligned}$$

$$2^3 + 2^6 + 2^7 = 8 + 64 + 128 = 200.$$

2. Fill in the squares of the grid with 9 consecutive integers so that the sums of the rows and columns are the numbers indicated in the margin of the table. The integers don't need to be consecutive in the table.

			47
			44
			35
35	42	49	+

You can start by determining which consecutive integers to use. The sum of the 9 integers must be $47+44+35$. If the string of consecutive integers starts at a then the sum is

$$a+(a+1)+\cdots+(a+8)=9a+36=126$$

So the numbers to use are

$$10, 11, 12, \cdots, 18.$$

10 must appear in the lower left square making the additional entries in the first column and last row $12, 13$ and $11, 14\ldots$

13	16	18	47
12	15	17	44
10	11	14	35
35	42	49	+

3. Every point on the plane is given one of two colors, red and blue. Show that there exists an isosceles triangle whose two equal sides have length 1 and whose vertices are of the same color.

Take any point in the plane and call it A. Let's say A is red. Draw a circle of radius 1 centered at A. If all points on the circle are blue, then fix any of them, say B, and take two points on the circle each a distance of 1 from B. That gives a desired isosceles triangle with three blue vertices.

If there's a red point on the first circle, say C, draw a new circle of radius 1 centered at C (note that this circle will pass through A). If there's a red point on the circle besides A, we get a desired isosceles triangle with three red vertices. If A is the only red point on that circle, then clearly there are three blue points on it, two of which are a distance 1 from C, which gives a triangle with blue vertices.

4. Peter was 10 the day before yesterday. Next year, he'll be 13. Explain how this is possible.

If today is 1 Jan 2024 then two days ago was 30 Dec 2023 and Peter was 10 years old. If his birthday is 31 Dec, then he turned 11 yesterday. This year, on 31 Dec 2024 he'll turn 12. And next year, in 2025, he'll be 13 on 31 Dec.

5. Show that for any positive integer n

$$\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}+\sqrt{n+1}}=\sqrt{n+1}-1.$$

Idea: rationalize denominator of each term in the sum.

$$\frac{1}{\sqrt{k}+\sqrt{k+1}} \times \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}} = \frac{\sqrt{k+1}-\sqrt{k}}{1}$$

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LEVEL II — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Suppose that $A \neq B$ and that

$$A^2 - B = 111$$
$$B^2 - A = 111$$

What could A and B be?

Subtracting the equations shows that $A^2 - B^2 + A - B = 0$. That says

$$(A-B)(A+B+1) = 0.$$

Since we're told $A \neq B$ we see that

$$A + B = -1.$$

Adding the equations shows that $A^2 + B^2 - (A+B) = 222$ so that

$$A^2 + B^2 = 221.$$

Remembering that $-A = B + 1$ we write

$$(B+1)^2 + B^2 = 221$$
$$2B^2 + 2B - 220 = 0$$
$$B^2 + B - 110 = 0$$
$$(B+11)(B-10) = 0$$

So $B = -11$ or $B = 10$ and the corresponding values of A are 10 and -11 respectively.

2. Mike and Katie are playing a game: given a number, the player can do one of the following four operations: multiply this number by 2, divide it by 2, multiply it by 3, or divide it by

3. The goal is to start with the number 12 and arrive at the number 54 in exactly 40 moves. Mike says it's impossible and Katie thinks it can be done. Who is right? Either present a solution or show that one does not exist.

Let m be the number of multiplications by 2, n the number of divisions by 2, j the number of multiplications by 3, and k the number of divisions by 3. We're looking for non-negative integers m, n, j, k such that

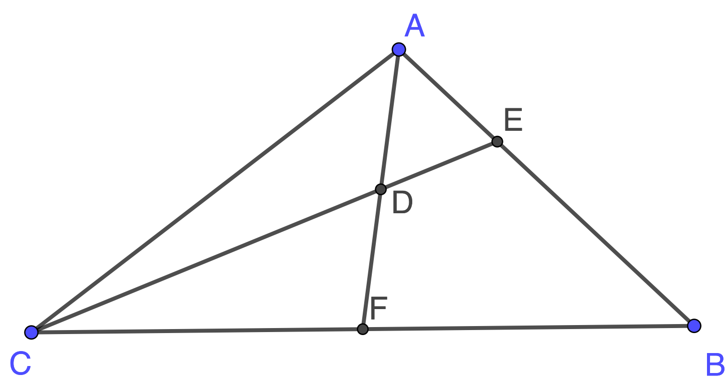
$$2^{m-n}3^{j-k}12 = 54, \quad \text{and} \quad m + n + j + k = 40$$

Since $12 = 2^2 \cdot 3$ and $54 = 2 \cdot 3^3$, the first equation can be rewritten as

$$2^{m-n+1}3^{j-k-2} = 1.$$

The only way this can happen is if $m - n + 1 = 0$ and $j - k - 2 = 0$. Then $m = n - 1$ and $j = k + 2$, so $m + n + j + k = 2n - 1 + 2k + 2 = 2n + 2k + 1$. This number is odd, so cannot be 40. $\boxed{\text{Mike is right.}}$

3. Let AF be the median of the triangle ABC, D be the midpoint of AF, and E be the point of intersection of the line CD with the side AB. If $BF = CF = BD$, prove that $AE = DE$. See the figure.



Triangle DBF is isosceles, so $\angle BFD = \angle BDF$. Therefore, $\angle ADB = \pi - \angle BDF = \pi - \angle BFD = \angle DFC$. Since, in addition, $AD = DF$ and $BF = FC$, we have $\Delta ADB = \Delta DFC$. This means that $\angle BAD = \angle CDF$. On the other hand, $\angle CDF = \angle EDA$, as vertical angles. Therefore, triangle EDA is isosceles, and, thus, $AE = DE$.

4. Concerning fractions:

(a) Show that for any two positive numbers a and b, we have $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$.

We have

$$\frac{a}{b}+\frac{b}{a}\geq 2 \quad\Longleftrightarrow\quad a^2+b^2\geq 2ab \quad\Longleftrightarrow\quad (a-b)^2\geq 0,$$

which is a true statement.

(b) Use the result of part (a) to show that for any three positive numbers a, b, and c, we have $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$.

We have

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right).$$

By part (a), each of the expressions in parentheses is no less than 2, so the sum will be no less than $3+2+2+2=9$.

5. After a teacher graded an exam, two students requested to take a make-up exam on different dates. After the first student took the make-up exam, getting a score of 80, the average score on the exam increased by 1. Three days later, the second student took the make-up exam. His score of 10 decreased the overall class average by 2. How many students took the exam originally?

If n is the number of students that originally took the exam and y is the original average then the effect of the first makeup exam says

$$\frac{ny+80}{n+1}=y+1$$

The effect of the second make up says

$$\frac{ny+80+10}{n+2}=y+1-2.$$

These two equations amount to

$$\begin{aligned} n+y &= 79\\ 2y-n &= 92.\end{aligned}$$

So $n=22$ and $y=57$.